8 Discrete Random Variables

D0 Intuitively, to tell whether a random variable is discrete, we simply consider the possible values of the random variable. If the random variable can be limited to only a finite or countably infinite number of possibilities, then it is discrete.

Example 8.1. Voice Lines: A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. [15, Ex 3-1]

Therefore, X in this example is a discrete RV.

D1 **Definition 8.2.** A random variable X is said to be a ***discrete random variable*** if there exists a countable number of distinct real numbers x_k such that

finite or countably infinite

$$\sum_k P[X = x_k] = 1. \tag{13}$$

D2 In other words, X is a discrete random variable if and only if X has a countable support.

Example 8.3. For the random variable N in Example 7.8 (Three Coin Tosses),

*H

The possible values are $0, 1, 2, 3$ (n_1, n_2, n_3, n_4)

The collection of possible values $\{0,1,2,3\}$ is finite. So the random variable is discrete.

For the random variable S in Example 7.9 (Sum of Two Dice),

The possible values are $2, 3, 4, \ldots, 12$ (s_1, s_2, s_3, s_{11})

The collection is finite so RV is discrete.

Example 8.4. Toss a coin until you get a H. Let N be the number of times that you have to toss the coin.

The possible values are $1, 2, 3, \ldots$

This collection is countably infinite.

RV is discrete.

Example 8.5. Measure the current room temperature.

The possible values are any real numbers between 273.15 to $\approx 1.417 \times 10^{32}$ °C. Any interval of positive length has uncountably many members in it. So, this random variable is <u>*not*</u> discrete.

8.6. Although the support S_X of a random variable X is defined as any set S such that $P[X \in S] = 1$. For discrete random variable, S_X is usually set to be $\{x : P[X = x] > 0\}$, the set of all "possible values" of X.

Definition 8.7. An ***integer-valued random variable*** is a discrete random variable whose x_k in (13) above are all integers.

8.8. Recall, from 7.21, that the ***probability distribution*** of a random variable X is a description of the probabilities associated with X. For a discrete random variable, the distribution can be described by just a list of all its possible values $(x_1, x_2, x_3, \ldots)$ along with the probability of each:

x	$P[X=x]$
x_1	$P[X=x_1]$
x_2	$P[X=x_2]$
$\vdots$	$\vdots$

$$(P[X = x_1], P[X = x_2], P[X = x_3], \ldots,).$$

In many cases, it is convenient to express the probability in the form of a formula. This is especially useful when dealing with a random variable that has infinite support. It would be tedious to list all the possible values and the corresponding probabilities.

8.1 PMF: Probability Mass Function

Definition 8.9. When X is a discrete random variable satisfying (13), we define its ***probability mass function*** (pmf) by[32]

$$p_X(x) = P[X = x].$$

- Sometimes, when we only deal with one random variable or when it is clear which random variable the pmf is associated with, we write $p(x)$ or p_x instead of $p_X(x)$.
- The argument (x) of a pmf ranges over *all real numbers*. Hence, the pmf is (and should be) defined for x that is not among the x_k in (13) as well. In such case, the pmf is simply 0. This is usually expressed as "$p_X(x) = 0$, otherwise" when we specify a pmf for a particular random variable.

[32] Many references (including [15] and `MATLAB`) does not distinguish the pmf from another function called the probability density function (pdf). These references use the function $f_X(x)$ to represent both pmf and pdf. We will *NOT* use $f_X(x)$ for pmf. Later, we will define $f_X(x)$ as a probability density function which will be used primarily for another type of random variable (continuous RV).

Additional Example:

In the in-class exercise, we roll a fair six-sided dice

$$\Omega = \{1, 2, 3, \ldots, 6\}$$

and define $Y(\omega) = 2 + (\omega-1)(\omega-3)(\omega-5)(\omega-7)$

ω	$Y(\omega)$
1	2
2	-13
3	2
4	11
5	2
6	-13

The possible values of Y are

$$\underbrace{-13,\ 2,\ 11}_{\text{three possibilities}}$$

$\downarrow$

finite

$\downarrow$

Y is discrete.

probability mass func. $p_Y(y) \equiv P[Y=y]$

pmf

When $y=2$, $p_Y(2) \equiv P[Y=2] = \frac{3}{6} = \frac{1}{2}$

$p_Y(1) \equiv P[Y=1] = 0$

$p_Y(11) \equiv P[Y=11] = P(\{4\}) = \frac{1}{6}$

$p_Y(-13) \equiv P[Y=-13] = P(\{2, 6\}) = \frac{2}{6} = \frac{1}{3}$

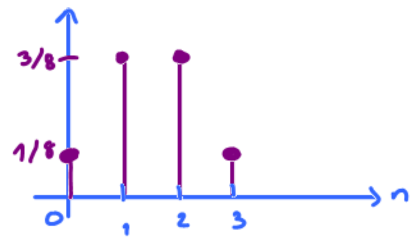
$$p_Y(y) = \begin{cases} 1/2, & y=2, \\ 1/6, & y=11, \\ 1/3, & y=-13, \\ 0, & \text{otherwise.} \end{cases}$$

- The pmf of a discrete random variable X is usually referred to as its ***distribution***.

Example 8.10. Continue from Example 7.8. N is the number of heads in a sequence of three coin tosses.

n	$P[N=n]$
0	1/8
1	3/8
2	3/8
3	1/8

pmf: $p_N(n) = \begin{cases} 1/8, & n = 0, 3, \\ 3/8, & n = 1, 2, \\ 0, & \text{otherwise.} \end{cases}$



8.11. Graphical Description of the Probability Distribution: Traditionally, we use ***stem plot*** to visualize p_X. To do this, we graph a pmf by marking on the horizontal axis each value with nonzero probability and drawing a vertical bar with length proportional to the probability.

8.12. Any pmf $p(\cdot)$ satisfies two properties:

(a) $p(\cdot) \geq 0$

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x.

When you are asked to verify that a function is a pmf, check these two properties.

8.13. Finding probability from pmf: for "any" subset B of $\mathbb{R}$, we can find

$$P[X \in B] = \sum_{x_k \in B} P[X = x_k] = \sum_{x_k \in B} p_X(x_k).$$

In particular, for integer-valued random variables,

$$P[X \in B] = \sum_{k \in B} P[X = k] = \sum_{k \in B} p_X(k).$$

8.14. Steps to find probability of the form P [some condition(s) on X] when the pmf $p_X(x)$ is known.

(a) Find the support of X.

(b) Consider only the x inside the support. Find all values of x that satisfy the condition(s).

(c) Evaluate the pmf at x found in the previous step.

(d) Add the pmf values from the previous step.

Example 8.15. Back to Example 7.7 where we roll one dice.

- The "important" probabilities are

$$P[X=1]=P[X=2]=\cdots=P[X=6]=\frac{1}{6}$$

- In tabular form:

Dummy variable

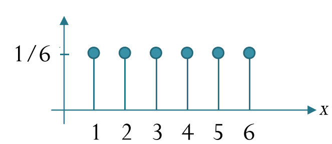
x	$P[X=x]$
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

- **Probability mass function (PMF)**:

$$p_X(x)=\begin{cases}1/6, & x=1,2,3,4,5,6,\\ 0, & \text{otherwise.}\end{cases}$$

- In general, $p_X(x)\equiv P[X=x]$

- Stem plot:

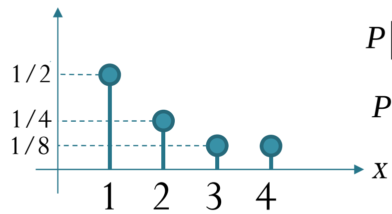


Suppose we want to find $P[X>4]$.

Steps	For this example…
Find the support of X.	The support of X is $\{1,2,3,4,5,6\}$.
Consider only the x inside the support. Find all values of x that satisfy the condition(s).	The members which satisfies the condition ">4" is 5 and 6.
Evaluate the pmf at x found in the previous step.	The pmf values at 5 and 6 are all 1/6.
Add the pmf values from the previous step.	Adding the pmf values gives $2/6=1/3$.

Example 8.16. Consider a RV X whose $p_X(x) = \begin{cases} 1/2, & x = 1, \\ 1/4, & x = 2, \\ 1/8, & x \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$

stem plot:



$$P[X = 2] = p_X(2) = \frac{1}{4}$$

$$P[X > 1] = p_X(2) + p_X(3) + p_X(4) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Example 8.17. Suppose a random variable X has pmf

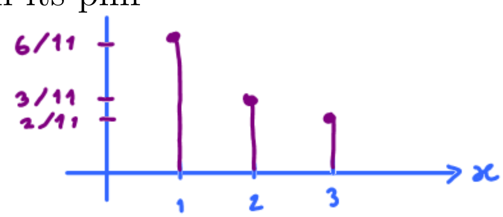
$$p_X(x) = \begin{cases} c/x, & x = 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

x	$p_X(x)$
1	$c/1 = c = 6/11$
2	$c/2 = 3/11$
3	$c/3 = 2/11$

(a) The value of the constant c is

"$\Sigma = 1$" : $c + \frac{c}{2} + \frac{c}{3} = 1 \Rightarrow c = \frac{6}{11}$

(b) Sketch its pmf



(c) $P[X = 1]$

$= p_X(1) = 6/11$

(d) $P[X \geq 2] = P[X = 2 \text{ or } 3] = p_X(2) + p_X(3) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}$

(e) $P[X > 3] = 0$

8.18. Any function $p(\cdot)$ on $\mathbb{R}$ which satisfies

(a) $p(\cdot) \geq 0$, and

(b) there exists numbers $x_1, x_2, x_3, \ldots$ such that $\sum_k p(x_k) = 1$ and $p(x) = 0$ for other x

is a pmf of some discrete random variable.

$p_X(x) \equiv P[X = x]$

8.2 CDF: Cumulative Distribution Function

Definition 8.19. The ***(cumulative) distribution function (cdf)*** of a random variable X is the function $F_X(x)$ defined by

$$F_X(x) = P[X \leq x].$$

- The argument (x) of a cdf ranges over all real numbers.
- From its definition, we know that $0 \leq F_X \leq 1$.
- Think of it as a function that collects the "probability mass" from $-\infty$ up to the point x.

8.20. From pmf to cdf: In general, for any discrete random variable with possible values $x_1, x_2, \ldots$, the cdf of X is given by

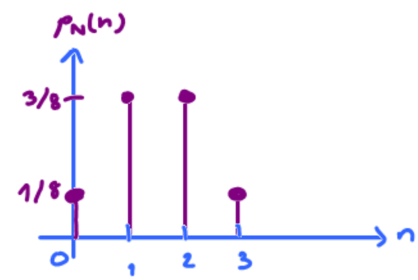
$$F_X(x) = P[X \leq x] = \sum_{x_k \leq x} p_X(x_k).$$

Example 8.21. Continue from Examples 7.8, 7.12, and 8.10 where N is defined as the number of heads in a sequence of three coin tosses. We have

$$p_N(0) = p_N(3) = \frac{1}{8} \text{ and } p_N(1) = p_N(2) = \frac{3}{8}.$$

cdf

(a) $F_N(0) \equiv P[N \leq 0] = P[N = 0] = p_N(0) = \frac{1}{8}$

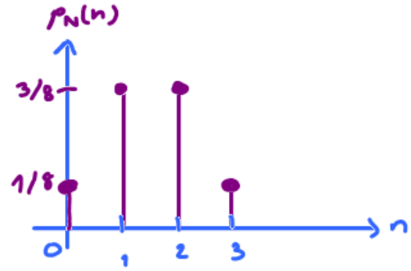


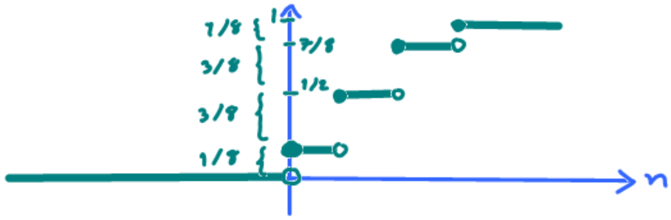
(b) $F_N(1.5) \equiv P[N \leq 1.5] = p_N(0) + p_N(1)$

$= \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$F_N(n)$

(c) Sketch of cdf





8.22. Facts:

- For any discrete r.v. X, F_X is a right-continuous, ***staircase*** function of x with jumps at a countable set of points x_k.
- When you are given the cdf of a discrete random variable, you can derive its pmf from the locations and sizes of the jumps. If a jump happens at $x = c$, then $p_X(c)$ is the same as the amount of jump at c. At the location x where there is no jump, $p_X(x) = 0$.

Example 8.23. Consider a discrete random variable X whose cdf $F_X(x)$ is shown in Figure 19.

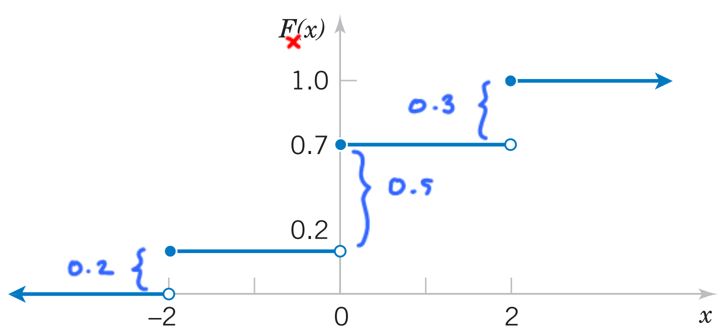
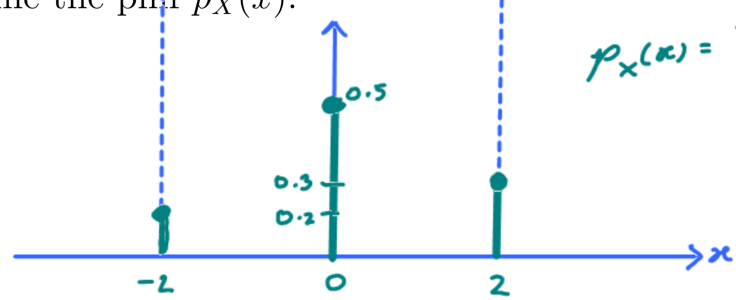


Figure 19: CDF for Example 8.23

Determine the pmf $p_X(x)$.



$$p_X(x) = \begin{cases} 0.2, & x = -2, \\ 0.5, & x = 0, \\ 0.3, & x = 2, \\ 0, & \text{otherwise}. \end{cases}$$

8.24. Characterizing[33] properties of cdf:

CDF1 F_X is non-decreasing (monotone increasing)

$\equiv$ if $a < b$, then $F_X(a) \leq F_X(b)$

CDF2 F_X is right-continuous (continuous from the right)

$\equiv$ for all x, $F_X(x^+) = F_X(x)$

$$F_X(c^+) \equiv \lim_{x \searrow c} F_X(x) \equiv \lim_{\substack{h \to 0 \\ h > 0}} F_X(x+h)$$

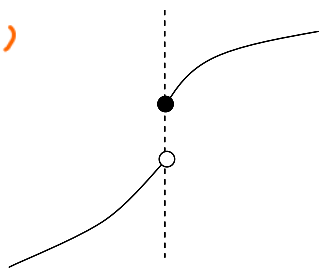


Figure 20: Right-continuous function at jump point

CDF3 $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$.

8.25. For discrete random variable, the cdf F_X can be written as

$$F_X(x) = \sum_{x_k} p_X(x_k) u(x - x_k),$$

where $u(x) = 1_{[0,\infty)}(x)$ is the unit step function.

[33] These properties hold for any type of random variables. Moreover, for any function F that satisfies these three properties, there exists a random variable X whose CDF is F.